IJMSHE Volume 2 Issue 1 | E-ISSN: 3082-3021 | DOI: https://doi.org/10.70847/593139

IJMSHE
International Journal of Multidisciplinary Studies in Higher Education

International Journal of Multidisciplinary Studies in Higher Education

Journal Homepage: https://ijmshe.com/index.php/pub/index

Research Article

Restrained Weakly Connected 2-Domination and the Lexicographic Product of Graphs

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Article Info

Article history:

Received: 31 December 2024
Revised: 22 January 2025
Accepted: 08 February 2025

Keywords:

Lexicographic product, Restrained weakly-connected 2-domination, Upper bound, Weakly connected

Abstract

Let $G = (V(G), E(G))$ be a nontrivial connected graph. A subset $S \subseteq V(G)$ is a restrained weakly connected 2-dominating set (RWC2D-set) in G if every vertex $v \in V(G)\backslash S$ is adjacent to at least two vertices in S and is adjacent to another vertex in $V(G)\backslash S$ and the subgraph $\langle S \rangle_w$ weakly induced by S is connected. A subgraph is said to be weakly induced by a set S if it includes all vertices in S and all edges in the original graph G that are connected to at least one vertex in S. The minimum cardinality of a restrained weakly connected 2-dominating set, denoted by $\gamma_{r2w}(G)$, is called the restrained weakly connected 2-domination number. Leveraging the concept of weakly connected 2-domination by examining another parameter called restrained domination is the prime focus of this study. The newly defined parameter is explored, establishing improved upper bounds and providing conditions for graph G to admit an RWC2D-set. Additionally, sufficient conditions for the RWC2D-set on the lexicographic product of graphs were provided. It is shown that for any graph G of order $n \geq 4$, $\gamma_{r2w}(G) \leq n - 2$. Moreover, the upper bound for the restrained weakly connected 2-domination number of the lexicographic product of graphs is provided.

Cite as: Tañalas, M., Militante, M., & Disca, B. (2025). Restrained Weakly Connected 2-Domination and the Lexicographic Product of Graphs. International Journal of Multidisciplinary Studies in Higher Education, 2(1), 138–149. https://doi.org/10.70847/593139

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Introduction

The study on domination in graph theory has enabled the modeling of systems and their relationships. With its various practical applications in communication networks, traffic networks, and security algorithms, among others, it has been extensively studied over the years. The concept of domination was formally introduced by Berge and Ore (1962), and since then, new types of domination parameters have arisen including 2-domination (Fink and Jacobson, 1985), (Maheswari et.al., 2021), weakly connected domination (Dunbar et.al., 1997), (Grossman, 1997), (Militante, 2021), and restrained domination (Domke et.al., 1999), (Militante and Eballe, 2022). The lexicographic product of graphs was first studied by Hausdorff (1914) and relevant theorems were established regarding domination in the lexicographic product of graphs as presented by Sandueta and Canoy (2014) and Militante and Eballe (2022). Several papers have been published regarding the above-mentioned parameters yet none has investigated the properties of the restrained weakly connected 2-domination (RWC2D) in the lexicographic product of graphs. For this reason, the said parameter is being pursued and explored. Published results from studies of Militante and Eballe (2022) provide foundations in conceptualizing this study. Some known special graphs such as wheel, star, complete, fan, and windmill graphs have established results in (Militante and Eballe, 2022). In this paper, we provide an improved upper bound as we consider another parameter, in particular, restrained domination.

A *wheel graph* $W_{1,n-1}$ is a graph of order n formed by connecting a single universal vertex called the axial vertex to all vertices of a cycle having $n-1$ vertices. A *star graph* $K_{1,n-1}$ is a graph in which $n-1$ vertices have degree 1 and a single vertex has degree $n-1$. A *complete graph* of order n, denoted by K_n, is a graph in which every two distinct vertices are adjacent. A *fan graph* $F_{1,n-1}$ is obtained by joining each vertex of a path of order $n-1$ to a single vertex. A *windmill graph* $W(m,k)$ is obtained by joining m copies of complete graphs K_k, where $k\geq 3$ at a common vertex. We also include some results for their respective complements. The *complement* of graph G, denoted by $\overline{G}$, is a graph with vertex set $V(\overline{G})$ such that the two vertices are adjacent in $\overline{G}$ if and only if these two vertices are not adjacent in G.

Throughout the paper, we consider G as a simple, finite, connected and undirected graph.

Preliminary Concepts

In this section, we provide definitions, concepts, and terminologies cited from (Sandueta and Canoy, 2014), (Militante and Eballe, 2022) that will be useful in the subsequent section.

Definition 2.1 (Sandueta and Canoy, 2014) The *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G)\times V(H)$ where $(u,u')(v,v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

A nonempty subset D of $V(G[H]) = V(G)\times V(H)$ can be expressed as $D = \bigcup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforward, we shall use the notation to denote any nonempty subset D of $V(G[H])$.

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Lemma 2.2 (Sandueta and Canoy, 2022) Let G and H be nontrivial connected graphs and let $D = \cup_{x \in S}\left(\{x\} \times T_x\right) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Then D is weakly connected in $G[H]$ if and only if S is weakly connected in G.

Definition 2.3 (Militante and Eballe, 2022) Let G be a nontrivial connected graph and S be a nonempty subset of $V(G)$. An *exclusive neighbor* of a vertex $x \in S$, relative to S is a vertex $y \in N_G(x) \cap (V(G)\setminus S)$ such that $N_G(y) \cap S = \{x\}$. The *set of exclusive neighbors* of a vertex x in G relative to S is given by $\{y \in V(G): y \in N_G(x) \cap (V(G)\setminus S),$ where $N_G(y) \cap S = \{x\}\}$.

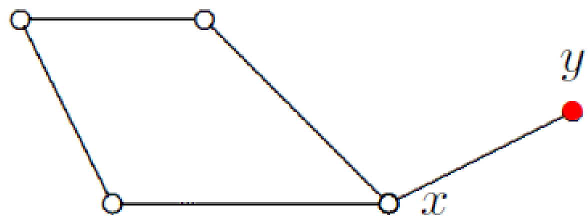

Figure 1. A connected graph G with vertex y as the exclusive neighbor of vertex x in G

Definition 2.4 (Militante and Eballe, 2022) Let $G = (V(G), E(G))$ be a graph. A set $C \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G)\setminus C$, there exists $u \in C$ such that $uv \in E(G)$, that is, $N_G[C] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set in G. Any dominating set in G whose cardinality is equal to $\gamma(G)$, is called a γ*-set* in G.

Definition 2.5 (Militante and Eballe, 2022) A subset S of $V(G)$ is a 2-*dominating set* in G if for every vertex $v \in V(G)\setminus S$, there exists at least two vertices in S adjacent to v, that is, $\left|N_G(v) \cap S\right| \geq 2$. The 2-*domination number*, $\gamma_2(G)$, of G is the smallest cardinality of a 2-dominating set in G. Any 2-dominating set in G whose cardinality is equal to $\gamma_2(G)$ is called a γ_2*-set* in G.

Definition 2.6 (Militante and Eballe, 2022) Let G be a connected graph. A dominating set $C \subseteq V(G)$ is called *weakly connected dominating set* in G if the subgraph $\langle C\rangle_w = \left(N_G[C], E_w\right)$ weakly induced by C is connected, where E_w is the set of all edges in G with at least one vertex in C. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set in G. Any weakly connected dominating set in G whose cardinality is equal to $\gamma_w(G)$ is called a γ_w*-set* in G.

Definition 2.7 (Militante and Eballe, 2022) Let G be a graph. A set $S \subseteq V(G)$ is a *total dominating set* in G if, for each $v \in V(G)$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G(S) = V(G)$. The *total domination number* $\gamma_t(G)$ of G is the smallest cardinality of a total dominating set in G.

Definition 2.8 (Militante and Eballe, 2022) Let G be a connected graph. A total dominating set S of $V(G)$ is a *weakly connected total dominating set* in G if the subgraph $\langle S\rangle_w = \left(N_G(S), E_w\right)$

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weakly induced by S is connected, where E_w is the set of all edges with at least one vertex in S. The *weakly connected total domination number* of G, denoted by $\gamma_{wt}(G)$, is the minimum cardinality among all weakly connected total dominating sets in G. Any weakly connected total dominating set in G whose cardinality is equal to $\gamma_{wt}(G)$ is called a γ_{wt}-*set* in G.

Definition 2.9 (Militante and Eballe, 2022) Let G be a nontrivial connected graph. A 2-dominating set $D \subseteq V(G)$ in G, is called a *weakly connected 2-dominating set* in G if the subgraph weakly induced by D is connected. The *weakly connected 2-domination number* of G, denoted by $\gamma_{2w}(G)$, is the smallest cardinality of a weakly connected 2-dominating set in G. A weakly connected 2-dominating set $D \subseteq V(G)$ with $|D| = \gamma_{2w}(G)$ is called a γ_{2w}-*set* in G.

Definition 2.10 (Militante and Eballe, 2022) Let G be a nontrivial connected graph. A weakly connected 2-dominating set $D \subseteq V(G)$, is called a *restrained weakly connected 2-dominating set* in G if every vertex in $V(G)\backslash D$ is adjacent to another vertex in $V(G)\backslash D$. The *restrained weakly connected 2-domination number* of G, denoted by $\gamma_{r2w}(G)$, is the smallest cardinality of a restrained weakly connected 2-dominating set in G. A restrained weakly connected 2-dominating set D with $|D| = \gamma_{r2w}(G)$ is called a γ_{r2w}-*set* in G.

To provide a comprehensive understanding of the concept of restrained weakly connected 2-domination, we include an illustrative example of the parameter.

Figure 2. Consider a graph $G = (\{a_1, a_2, a_3, a_4\}, \{a_1a_2, a_1a_4, a_2a_3, a_2a_4, a_3a_4\})$ in Figure 2 (a). Suppose $S = \{a_1, a_3\} \subseteq V(G)$. Then S is a dominating set. We examine the said graph according to the conditions of restrained weakly connected 2-domination. It can be observed that every element in $V(G)\backslash S = \{a_2, a_4\} \subseteq V(G)$ is 2-dominated by elements in S. Hence, S is a 2-dominating set. Further, as shown in Figure 2 (b), the subgraph weakly induced by S, $\langle S\rangle_w$, is isomorphic to C_4, and is connected, implying that the neighborhood of each element in $V(G)\backslash S$ contains at least one element in S. Now, the subgraph induced by $\langle V(G)\backslash S\rangle$ is isomorphic to path P_2, hence it has no isolated vertex as shown in Figure 2 (c). The three results suggest that the given graph G admits a restrained weakly connected 2-dominating set. Hence, $\gamma_{r2w}(G) \leq |S| = 2$. Since there is no RWC2D-set of cardinality 1, we conclude $\gamma_{r2w}(G) = 2$.

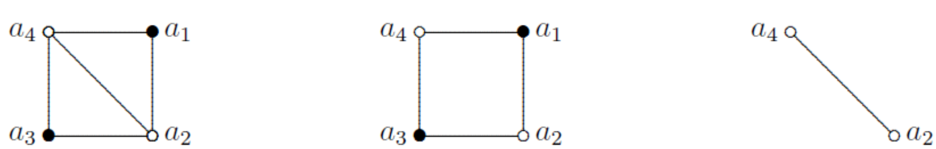

Figure 2: (a) A graph of order 4; (b) the subgraph weakly induced by S; and (c) the subgraph induced by V(G)\S with colored vertices representing S

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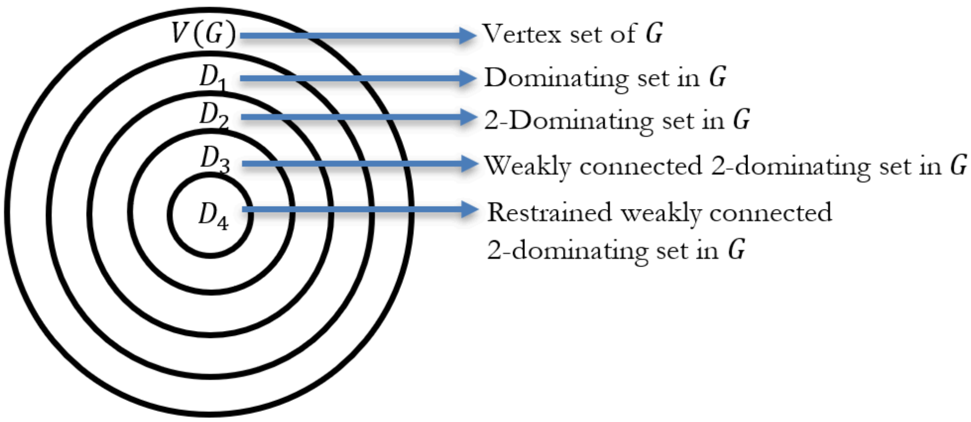

Figure 3: Relation between vertex set, dominating set, 2-dominating set, weakly connected 2-dominating set, and the restrained weakly connected 2-dominating set.

The relationship of the aforementioned domination parameters can be represented using the diagram below. Within the vertex set of G, $V(G)$, is the dominating set D_1 which contains the 2-dominating set D_2 which contains the weakly connected 2-dominating set D_3 which contains restrained weakly connected 2-dominating set D_4.

Results

In this section, we present some bounds of the parameter and the sufficient conditions for RWC2D-set in the lexicographic product of graphs.

Remark 3.1 *Let G be a connected graph of order $n \geq 4$. Then $2 \leq \gamma_2(G) \leq \gamma_{2w}(G) \leq \gamma_{r2w}(G) \leq n$.*

Remark 3.2 *Let G be any nontrivial connected graph. Then the leaves of G belong to any RWC2D-set in G.* The rightmost upper bound given in Remark 3.1 can be improved for some graphs of order $n \geq 4$. This is formally stated in Theorem 3.3.

Theorem 3.3 *Let G be a connected graph of order $n \geq 4$. Then G has a proper RWC2D subset if and only if $\gamma_{r2w}(G) \leq n - 2$.*

Proof: Let $D \subsetneq V(G)$ be a restrained weakly connected 2-dominating set in G. Then $V(G) \setminus D \neq \emptyset$. Let $v \in V(G) \setminus D$. Since D is a restrained dominating set, $\langle V(G) \setminus D \rangle$ has no isolated vertex, that is, there exists $w \in V(G) \setminus D$ such that $vw \in E(G)$. Therefore, $\gamma_{r2w}(G) \leq |D| = n - 2$.

The converse is immediate. ■

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Theorem 3.4 *Let G be a connected graph of order $n \geq 4$. If $D \subseteq V(G)$ is a restrained weakly connected 2-dominating set in G, then for every $v \in V(G) \setminus D$, $|N_G(v) \cap D| \geq 2$ and $|N_G(v) \cap (V(G) \setminus D)| \geq 1$.*

Theorem 3.5 *Let G be a connected graph of order $n \geq 4$ and let D be a restrained weakly connected 2-dominating set in G. If $v \in V(G) \setminus D$, then $deg_G(v) \geq 3$.*

Proof: Let D be a restrained weakly connected 2-dominating set in G and let $v \in V(G) \setminus D$. Since D is a restrained 2-dominating set, there exist $u, w \in D$ such that $uv, wv \in E(G)$ and $x \in V(G) \setminus D$ such that $vx \in E(G)$. It follows that $\{u, w, x\} \subseteq N_G(v)$. Therefore, $deg_G(v) \geq 3$. ■

The restrained weakly connected 2-dominating set occurs only when a graph is connected. Generally, it is possible to observe that the complement $\overline{G}$ of a graph G admits a restrained weakly connected 2-dominating set if G has no spanning complete bipartite subgraph. The following theorem shows a specific instance in which the set does not exist. We refer to $\Delta(G)$ the maximum degree of G.

Theorem 3.6 *Let G be any connected graph of order $n \geq 4$. If $\Delta(G) = n - 1$, then the complement $\overline{G}$ has no restrained weakly connected 2-dominating set.*

Proof: If $deg_G(v) = n - 1$, then $deg_{\overline{G}}(v) = 0$. It means that v is an isolated vertex of $\overline{G}$. Thus, $\overline{G}$ is a disconnected graph. Therefore, $\overline{G}$ does not admit a restrained weakly connected 2-dominating set. ■

Corollary 3.7 *Let $n \geq 4$. Then the complements $\overline{W}_{1,n-1}$, $\overline{K}_{1,n-1}$, $\overline{K}_n$, $\overline{F}_{1,n-1}$ and $\overline{W}(m, k)$ The wheel graph, star graph, complete graph, fan graph and windmill graph, respectively, have no restrained weakly connected 2-dominating sets.*

Remark 3.8 *If G and H are two graphs with G connected, then $G[H]$ is connected.*

Remark 3.8 assures that given a connected graph G and any graph H, the lexicographic product $G[H]$ is connected. Hence, $G[H]$ admits a restrained weakly connected 2-dominating set whenever G is a connected graph. The next theorem provides sufficient conditions for a restrained weakly connected 2-dominating set in $G[H]$.

Theorem 3.9 *Let G and H be nontrivial connected graphs. Then $D = \cup_{x \in S} [\{x\} \times T_x] \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a restrained weakly connected 2-dominating set in $G[H]$ if S is a weakly connected total 2-dominating set in G with $|T_x| = 2$ for every $x \in S$; or $S \subseteq V(G)$ is a weakly connected dominating set in G satisfying the following properties:*

- *For every vertex $x \in S$ that has an exclusive neighbor, $|T_x| \geq 2$;*
- *For every $x \in S \setminus N_G(S)$, T_x is a 2-dominating set in H;*
    - *For every $x \in S$ with $|N_G(x) \cap S| = 1$, one of the following holds:*
    - *T_x is a restrained dominating set;*

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- ○ *T_x is a dominating set in H and $\left|V(H)\backslash T_y\right| \geq 1$, for some $y \in N_G(x) \cap S$;*
- ○ *T_x is a dominating set in H and $N_G(x) \cap (V(G)\backslash S) \neq \emptyset$;*
- ○ *$\left|T_y\right| \geq 2$ and $\langle V(H)\backslash T_x\rangle$ has no isolated vertex where $y \in N_G(x) \cap S$;*
- ○ *$\left|T_y\right| \geq 2$ and $\left|V(H)\backslash T_y\right| \geq 1$ where $y \in N_G(x) \cap S$;*
- ○ *$\left|T_y\right| \geq 2$ and $N_G(x) \cap (V(G)\backslash S) \neq \emptyset$ where $y \in N_G(x) \cap S$;*

- *For every $x \in S$, with $\left|N_G(x) \cap S\right| \geq 2$, either $\langle V(H)\backslash T_x\rangle$ has no isolated vertex or $\left|V(H)\backslash T_y\right| \geq 1$ for some $y \in N_G(x) \cap S$.*

Proof: Let S be a weakly connected total dominating set in G with $\left|T_x\right| = 2$ for every $x \in S$. Then by Lemma 2.2, $D = \cup_{x\in S}\left(\{x\} \times T_x\right) \subseteq V(G[H])$ is weakly connected in $G[H]$. Since $\left|T_x\right| = 2$ and S is a total dominating set in G, every vertex $v \in V(G[H])\backslash D$ is 2-dominated by T_x. Now, let $p \in V(G[H] \setminus D)$. Then $p \in V(H)\setminus T_x$ for $x \in S$, or $p \in T_z$ for some $z \in V(G)\backslash S$. By adjacency of vertices in $G[H]$ and by the fact that both G and H are nontrivial connected graphs, there exists $q \in V(G[H])\backslash D$ such that $pq \in E(G[H])$, where either $q \in V(H)\backslash T_x$ or $q \in T_w$ with $w \in V(G)\backslash S$ and $xw \in E(G)$. Hence, D is a restrained weakly connected 2-dominating set in $G[H]$.

Next, suppose that S is a weakly connected dominating set in G satisfying the properties (a), (b), (c), and (d) as stated in the theorem. Then by Lemma 2.2, $D = \cup_{x\in S}\left(\{x\} \times T_x\right) \subseteq V(G[H])$ is weakly connected in $G[H]$. Let $v \in V(G[H])\backslash D$. Then either $v \in T_x$ for some $x \in S$ or $v \in T_{x'}$ for some $x' \in V(G)$ The statements (a), (b), (c) and (d) of the theorem together with the property of the set S will guarantee that $\left|N_{G[H]}(v) \cap D\right| \geq 2$.

Next, to show that $\langle V(G[H])\backslash D\rangle$ has no isolated vertex, consider the following cases:

Case 1. Suppose $x \in V(G)\backslash S$. Then x is either an exclusive neighbor or not an exclusive neighbor relative to the assigned weakly connected dominating set S. In either case, $\langle\{x\}\times V(H)\rangle$ has no isolated vertex since H is connected and nontrivial.

Case 2. Suppose $x \in S$. If $T_x = V(H)$, then we are done. Suppose that $T_x \subsetneq V(H)$, we consider the following cases:

Subcase 2.1 Suppose $x \in S\backslash N_G(S)$. Let $a \in V(H)\backslash T_x$. Then $(x, a) \notin D$. Since G is a nontrivial connected graph, there exists $y \in V(G)$ such that $xy \in E(G)$. Since $x \in S\backslash N_G(S)$, we have $y \in V(G)\backslash S$. Choose $b \in V(H)$. Then $(y, b) \notin D$. Since $xy \in E(G)$, we get $(x, a)(y, b) \in E(G[H])$.

Subcase 2.2 Suppose that $\left|N_G(x) \cap S\right| = 1$. Let $x \in V(H)\backslash T_x$ and $a \in V(H)$. Then $(x, a) \in V(G[H])\backslash D$. By property (c) ((i) up to (vi)), as stated in the theorem, either there exists $(y, b) \in V(G[H])\backslash D$ such that $(x, a)(y, b) \in E(G[H])$ for some $y \in V(G) \cap S$; or there exists $(z, b') \in V(G[H])\backslash D$ such that $(x, a)(z, b') \in E(G[H])$ for some $z \in V(G)\backslash S$

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Subcase 2.3 Suppose that $\left|N_G(x) \cap S\right| \geq 2$. This means that $|V(G)| \geq 3$. Let $a \in V(H)\backslash T_x$. Then $(x, a) \in V(G[H])\backslash D$. By property (d) as stated in the theorem, either $ab \in E(H)$ for some $b \in V(H)\backslash T_x$ or $V(H)\backslash T_y \neq \emptyset$ for some $y \in N_G(x) \cap S$ with $xy \in E(G)$. It follows that $(x, a)(x, b) \in E(G[H])$ where $(x, b) \in V(G[H])\backslash D$ or $(x, a)(y, c) \in E(G[H])$ where $(y, c) \in V(G[H])\backslash D$ for some $c \in V(H)\backslash T_y$.

In all cases the subgraph $\langle V(G)\backslash D\rangle$ has no isolated vertex. Accordingly, $D \subseteq V(G[H])$ is a RWC2D-set. ■

The next result is a direct consequence of Theorem 3.9.

Let G and H be nontrivial connected graphs. Then $\gamma_{r2w}(G[H]) \leq min\{2 \cdot \gamma_{wt}(G), \gamma_w(G) \cdot \gamma_2(H)\}$.

Proof: Let C_1 and C_2 be γ_{wt}-set and γ_w-set in G, respectively. Let $C_1^{'} \subseteq V(H)$ such that $\left|C_1^{'}\right| = 2$ and $C_2^{'}$ be a γ_2-set in H. Consider $T_x = C_1^{'}$ for every $x \in C_1$ and $T_x = C_2^{'}$ for every $x \in C_2$. Then by Theorem 3.9, $D = \cup_{x \in C_1}\left(\{x\} \times C_1^{'}\right) \subseteq C_1 \times C_1^{'}$ is a restrained weakly connected 2-dominating set in $G[H]$. It follows that $\gamma_{r2w}(G[H]) \leq |D| = \left|C_1\right|\left|C_1^{'}\right| = 2 \cdot \gamma_{wt}(G)$. Similarly, by Theorem 3.9, $D^{'} = \cup_{x \in C_2}\left(\{x\} \times T_x\right) \subseteq C_2 \times C_2^{'}$ is a restrained weakly connected 2-dominating set in $G[H]$. Thus, $\gamma_{r2w}(G[H]) \leq \left|D^{'}\right| = \Sigma_{x \in C_2}\left|T_x\right| \leq \left|C_2\right|\left|C_2^{'}\right| = \gamma_w(G) \cdot \gamma_2(H)$. ■

The succeeding example will illustrate that the bound given in corollary 3.10 is sharp.

Figure 4. Consider the lexicographic products $P_4\left[P_4\right]$ and $P_3\left[P_3\right]$ in Figure 4. In Figure 4 (a), $\gamma_{r2w}\left(P_4\left[P_4\right]\right) = 4$ while in Figure 4 (b), we have $\gamma_{r2w}\left(P_3\left[P_3\right]\right) = 2$. These graphs show that the bound given in Corollary 3.10 is sharp.

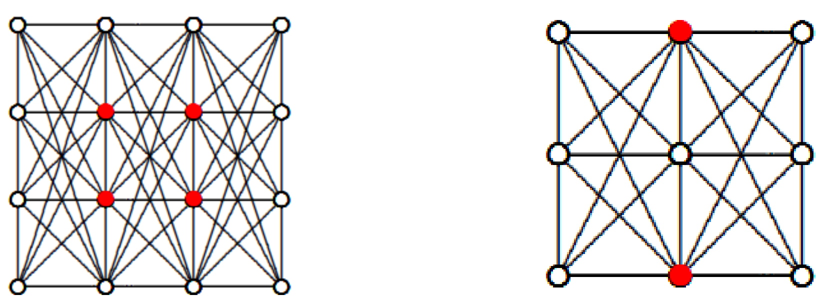

Figure 4. (a) The graphs $P_4[P_4]$ and $P_3[P_3]$ (b) with colored vertices in some RWC2D-sets.

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Now, we present a characterization for the restrained weakly connected 2-domination number equal to 2.

Theorem 3.12 *Let G and H be two nontrivial connected graphs. Then* $\gamma_{r2w}(G[H]) = 2$ *if and only if one of the following holds:*

- $\gamma(G) = 1$ *and* $\gamma_2(H) = 2$.
- $G = K_n$, $n \geq 3$ *and* $\gamma(H) = 1$.

Proof: Suppose that $\gamma_{r2w}(G[H]) = 2$. Let $D \subseteq V(G[H])$ be a γ_{r2w}-set. Then D can be expressed as $D = \cup_{x\in S}(\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$ and $|D| = 2$. There are two possibilities for S, namely, $|S| = 1$ or $|S| = 2$. Suppose $|S| = 1$, say $S = \{x\}$. Then $|T_x| = 2$. Since D is a weakly connected dominating set in $G[H]$, by Lemma 2.2, S is a weakly connected dominating set in G. It follows that S is a dominating set in G. Also, since D is a 2-dominating set in $G[H]$ with $|T_x| = 2$, we have $\gamma_2(H) = 2$. As a consequence, we have $\gamma(G) = 1$ and $\gamma_2(H) = 2$. Next, suppose $|S| = 2$, say $S = \{x, y\}$. Then $|T_x| = |T_y| = 1$. Since D is a 2-dominating set, the vertices in G must be pairwise adjacent, and T_x must be a dominating set in H. Hence, G is a complete graph of order at least two and that $\gamma(H) = 1$.

Conversely, suppose first that $\gamma(G) = 1$ and $\gamma_2(H) = 2$. Let $S = \{x\}$ be a γ-set in G and C be a γ_2-set in H. Then $S = \{x\}$ is a weakly connected dominating set in G. By Theorem 3.9, $D = \cup_{x\in S}(\{x\} \times T_x) \subseteq S\times C$ is a restrained weakly connected 2 - dominating set in $G[H]$. Hence, $\gamma_{r2w}(G[H]) \leq |D| = |S||C| = 1 \cdot \gamma_2(H) = 2$. By Remark 3.1, we get $\gamma_{r2w}(G[H]) = 2$. Now, suppose that $G = K_n$, $n \geq 3$ and $\gamma(H) = 1$. Pick $S = \{x, y\}$ where x, $y \in V(Kn)$. Then S is a weakly connected total 2-dominating set in K_n. Let $T_x = C_1$ be a γ-set in H with $|C_1| = 1$. Then by Theorem 3.9, $D = \cup_{x\in S}(\{x\} \times T_x) \subseteq S\times C_1$ is a restrained weakly connected 2-dominating set in $G[H]$. Thus, $\gamma_{r2w}(G[H]) \leq |D| = |S||C_1| = 2$. By Remark 3.1, we have $\gamma_{r2w}(G[H]) = 2$. ■

The next result presents a specific case of the parameter's value when the graph G achieves the smallest domination number. For clarity, this is illustrated in Example 3.14.

Remark 3.13 *If G and H are nontrivial connected graphs with* $\gamma(G) = 1$. *Then the following can be observed:*

- $2 \leq \gamma_{r2w}(G[H]) \leq 4$.
- *If* $\gamma_2(H) = 2$, *then* $\gamma_{r2w}(G[H]) = 2$.
- *If* $\gamma(H) = 2$ *and* $\gamma_2(H) \neq 2$, *then* $\gamma_{r2w}(G[H]) = 3$.
- *If* $\gamma(H) = 3$ *and* $\gamma_2(H) \neq 3$, *then* $\gamma_{r2w}(G[H]) = 4$.

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Figure 5. Consider the lexicographic product $K_{1,3}[C_4]$ of star graph $K_{1,3}$ and cycle C_4 in Figure 5. The figure shows the minimum restrained weakly connected 2-dominating set (colored vertices). Moreover, $\gamma_{r2w}(K_{1,3}[C_4]) = \gamma_{2w}(C_4) = 2$.

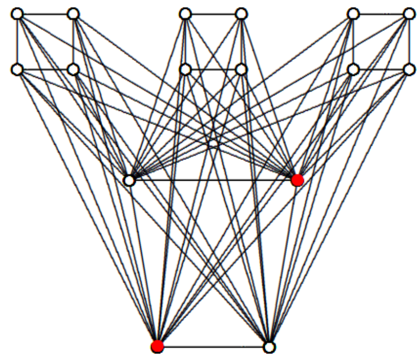

Figure 5. The lexicographic product with colored vertices in some RWC2D-sets.

Remark 3.15 *Let H be any nontrivial connected graph and P_2 be the path of order 2. Then:*

- $\gamma_{r2w}(P_2[H]) = 2$ *if and only if either* $\gamma_2(H) = 2$ *or* $\gamma(H) = 1$.
- $\gamma_{r2w}(P_2[H]) = 3$ *if and only if* $\gamma_{r2w}(P_2[H]) \neq 2$ *and either* $\gamma_2(H) = 3$ *or* $\gamma(H) = 2$.
- $\gamma_{r2w}(P_2[H]) = 4$ *if and only if* $\gamma_{r2w}(P_2[H]) \neq 3$ *and either* $\gamma_2(H) = 4$, *or* $\gamma(H) = 3$, *or* $\gamma_2(H) > 4$, *or* $\gamma(H) > 3$.

Conclusion

This study investigated the bounds and conditions of the newly defined parameter, providing improved upper bounds and establishing conditions under which G admits a RWC2D-set. The analysis is also extended to the lexicographic product of graphs, identifying sufficient conditions for the existence of such sets. These findings contribute to the broader understanding of domination in graphs, with potential implications for network theory and related fields.

It is recommended to extend this study by exploring the bounds of the aforementioned parameter for other graph products, including the Cartesian, tensor, strong, and co-normal products of graphs, in addition to the lexicographic product examined here.

Acknowledgment: The authors acknowledge the support of their institution, Mindanao State University – General Santos.

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